

A Generalization to Cantor Series of Sondow's Geometric Proof that e Is Irrational and His Measure of Its Irrationality

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Abstract

Using the geometric idea, due J. Sondow, and encouraged by himself, we give a geometric proof for Cantor's Theorem. Moreover, it is given a irrationality measure for some Cantor series.

Key words: Irrationality, irrationality measure, Cantor, Smarandache function.

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1 Introduction

In 2006, Jonathan Sondow gave a nice geometric proof that e is irrational. Moreover, he said that a generalization of his construction may be used to prove the Cantor's theorem. But, he didn't do it in his paper, see [1]. So, this work will give a geometric proof to Cantor's theorem using Sondow's construction. After, it is given an irrationality measure to some Cantor series, for that, we generalize the Smarandache function. Also we give a irrationality measure for e that is a slight improvement the given one in [1]. Finally, we discuss the existence of geometric irrationality proofs to irrational numbers.

2 Cantor's Theorem

We start with a definition,

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Definition 1 Let $a_0, a_1, \dots, b_1, b_2, \dots$ be sequences of integers that satisfy the inequalities $b_n \geq 2$, and $0 \leq a_n \leq b_n - 1$ if $n \geq 1$. Then the convergent series

$$\theta := a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \dots \quad (1)$$

is called *Cantor series*.

Example 1 The number e is a Cantor series. For see that, take $a_0 = 2, a_n = 1, b_n = n + 1$ for $n \geq 1$.

We recall the following statement,

Theorem 1 (Cantor) Let θ be a Cantor series. Suppose that each prime divides infinitely many of the b_n . Then θ is irrational if and only if both $a_n > 0$ and $a_n < b_n - 1$ hold infinitely often.

Proof For proving the necessary condition, suppose on the contrary, that is, either $a_n > 0$ or $a_n < b_n - 1$ doesn't happen infinitely often. If the first case happens, there exist $n_0 \in \mathbb{N}$ such that $a_n = 0$, for all $n \geq n_0$, then clearly θ is a rational number. If the second case happens, there exist $n_0 \in \mathbb{N}$, with $a_n = b_n - 1$, if $n \geq n_0$. After a simple calculation, we have $\theta = a_0 + \frac{a_1}{b_1} + \dots + \frac{a_{n_0-1}+1}{b_1 \dots b_{n_0-1}} \in \mathbb{Q}$. For showing the sufficient condition, we going to construct a nested sequence of closed intervals I_n with intersection θ . Let $I_1 = [a_0 + \frac{a_1}{b_1}, a_0 + \frac{a_1+1}{b_1}]$. Proceeding inductively, we have two possibilities, the first one, if $a_n = 0$, so define $I_n = I_{n-1}$. When $a_n \neq 0$, divide the interval I_{n-1} into $b_n - a_n + 1$ (≥ 2) subintervals, the first one with length $\frac{a_n}{b_1 \dots b_n}$ and the other ones with equal length, namely, $\frac{1}{b_1 \dots b_n}$, and let the first one be I_n . By construction, $|I_n| \geq \frac{1}{b_1 \dots b_n}$, for all $n \in \mathbb{N}$ and when $a_n \neq 0$, the length of I_n is exactly $\frac{1}{b_1 \dots b_n}$. By hypothesis on a_n , there exist infinitely many $n \in \mathbb{N}$, such that $|I_n| = \frac{1}{b_1 \dots b_n}$. Thus, we have

$$I_n = \left[a_0 + \frac{a_1}{b_1} + \dots + \frac{a_n}{b_1 \dots b_n}, a_0 + \frac{a_1}{b_1} + \dots + \frac{a_n+1}{b_1 \dots b_n} \right] = \left[\frac{A_n}{b_1 \dots b_n}, \frac{A_n+1}{b_1 \dots b_n} \right]$$

where $A_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$. Also $\theta \in I_n$ for all $n \geq 1$ (*proof*. By hypothesis, it is easy see that $\theta > \frac{A_n}{b_1 \dots b_n}$, for all $n \geq 1$. For the other inequality, note that $\frac{a_m}{b_m} \leq 1 - \frac{1}{b_m}$, for all $m \in \mathbb{N}$, therefore

$$b_1 \dots b_n (\theta - (a_0 + \frac{a_1}{b_1} + \dots + \frac{a_n}{b_1 \dots b_n})) \leq 1 \quad (2)$$

Moreover, if $a_m < b_m - 1$, then holds the strict inequality in (2), for each $n < m$). Since $a_n > 0$ holds infinitely often,

$$\bigcap_{n=1}^{\infty} I_n = \theta.$$

Suppose that $\theta = \frac{p}{q} \in \mathbb{Q}$. Each prime number divides infinitely many b_n , so there exist n_0 sufficiently large such that $q|b_1 \cdots b_{n_0}$ and $a_{n_0} \neq 0$. Hence $b_1 \cdots b_{n_0} = kq$ for some $k \in \mathbb{N}$. Take $N \geq n_0$, such that, $a_{N+1} < b_{N+1} - 1$. Hence θ lies in interior of I_N . Also $I_N = I_{n_0+k}$ for some $k \geq 0$. Suppose $I_N = I_{n_0}$. We can write $\theta = \frac{kp}{b_1 \cdots b_{n_0}}$, thus $\frac{A_{n_0}}{b_1 \cdots b_{n_0}} < \frac{kp}{b_1 \cdots b_{n_0}} < \frac{A_{n_0}+1}{b_1 \cdots b_{n_0}}$. But that is an absurd. If $I_N = I_{n_0+k}$, for $k \geq 1$, then we write $\theta = \frac{kpb_{n_0+1} \cdots b_{n_0+k}}{b_1 \cdots b_{n_0+k}}$. But that is an absurd too. Therefore, follows the irrationality of θ . \square

3 Irrationality measure

The next step is to give a irrationality measure for some Cantor series. Now, we construct a non-countable family of functions, where one of them is exactly a well-known function for us.

Definition 2 Given $\sigma = (b_1, b_2, \dots) \in \mathbb{N}^\infty$, where

(i) For all p prime number, $\#\{n \in \mathbb{N} \mid p|b_n\} = \infty$.

We define the function $D(\cdot, \sigma) : \mathbb{Z}^* \rightarrow \mathbb{N}$, by

$$D(q, \sigma) := \min\{n \in \mathbb{N} \mid q|b_1 \cdots b_n\}$$

Note that $D(\cdot, \sigma)$ is well defined, by condition (i) and the well-ordering theorem.

In [1], J. Sondow showed that for all integers p and q with $q > 1$,

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q) + 1)!}, \quad (3)$$

where $S(q)$ is the smallest positive integer such that $S(q)!$ is a multiple of q (the so-called Smarandache function, see [2]). Note that if $\eta = (1, 2, 3, \dots)$, then $D(q, \eta) = S(q)$, for $q \neq 0$. Since e is a Cantor series and $D(\cdot, \sigma)$ is a generalization of Smarandache function, it is natural to think in a generalization or an improvement to (3).

Let θ be a Cantor series. Define, for each $n \geq 1$, the number θ_n as

$$\theta_n = \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+1}b_{n+2}} + \dots$$

Note the, $\theta_n \leq 1$, if $n \geq 1$, see (2). With the same notations of the proof of Theorem 1, we have the following result

Lemma 1 *If $\theta_n \leq \frac{1}{2}$, then*

$$\left| \theta - \frac{m}{b_1 \cdots b_n} \right| \geq \left| \theta - \frac{A_n}{b_1 \cdots b_n} \right| \quad (4)$$

for all $m \in \mathbb{Z}$.

Proof First, we want to prove for $m = A_n + 1$. Suppose on the contrary, that is, $\theta - \frac{A_n}{b_1 \cdots b_n} > \frac{A_n+1}{b_1 \cdots b_n} - \theta$. Therefore $\theta_n = b_1 \cdots b_n (\theta - (a_0 + \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_1 \cdots b_n})) > \frac{1}{2}$. Contradiction. Suppose now that there exist $m \in \mathbb{Z}$ such that (4) is not true. Using the case above, we have

$$\left| \theta - \frac{m}{b_1 \cdots b_n} \right| < \left| \theta - \frac{A_n}{b_1 \cdots b_n} \right| \leq \left| \theta - \frac{A_n+1}{b_1 \cdots b_n} \right|$$

So, $\frac{m}{b_1 \cdots b_n}$ lies in interior of I_n . Contradiction. Hence (4) holds for all $m \in \mathbb{Z}$. \square

A sufficient condition for $\theta_n \leq \frac{1}{2}$, for all n , is that $4a_m \leq b_m$, for each $m \geq 2$. The next result gives an irrationality measure for some Cantor series.

Proposition 1 *Suppose that a Cantor series θ , satisfying (i), is irrational and $\theta_n \leq \frac{1}{2}$ for $n \geq 2$, then for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$, with $D(q, \sigma) > 1$,*

$$\left| \theta - \frac{p}{q} \right| > \frac{a_{D(q, \sigma)+1}}{b_1 \cdots b_{D(q, \sigma)+1}} \quad (5)$$

where $\sigma = (b_1, b_2, \dots)$.

Proof Let $\sigma = (b_1, b_2, \dots)$. Set $n = D(q, \sigma)$ and $m = \frac{pb_1 \cdots b_n}{q}$. Therefore m, n are integers. If $a_{n+1} = 0$, then (5) follows. If $a_{n+1} \neq 0$ and using the Lemma 1,

$$\left| \theta - \frac{p}{q} \right| = \left| \theta - \frac{m}{b_1 \cdots b_n} \right| > \frac{a_{n+1}}{b_1 \cdots b_{n+1}}$$

That is the desired result. \square

The result below gives a slight improvement to (3).

Corollary 1 *If p and q are integers, with $q \neq 0$, then*

$$\left| e - \frac{p}{q} \right| > \frac{1}{(D(q, \sigma) + 2)!}, \quad (6)$$

where $\sigma = (2, 3, 4, \dots)$

Proof Since $\min_{p \in \mathbb{Z}} |e - p| > 0.28 > \frac{1}{6}$, so (6) holds in the case $q = \pm 1$. When $q \neq \pm 1$, (6) holds by Proposition 1 and Example 1. Moreover, in this case, we

have $S(q) - 1 \in \{n \in \mathbb{N} \mid q|(n+1)!\}$ and $D(q, \sigma) + 1 \in \{n \in \mathbb{N} \mid q|n!\}$. Thus $S(q) = D(q, \sigma) + 1$. So

$$\left| e - \frac{p}{q} \right| > \frac{1}{(D(q, \sigma) + 2)!} = \frac{1}{(S(q) + 1)!}$$

□

Actually, the improvement happens only because (6) also holds for $q = \pm 1$.

Example 2 *The number $\xi := \frac{1}{(1!)^5} + \frac{1}{(2!)^5} + \frac{1}{(3!)^5} + \dots = 1.031378\dots$ is irrational, moreover for $p, q \in \mathbb{Z}, q \neq 0$, holds*

$$\left| \xi - \frac{p}{q} \right| > \frac{1}{(D(q, \sigma) + 2)!^5}$$

where $\sigma = (2^5, 3^5, \dots)$.

We finish with an interesting conditional result.

Corollary 2 *If θ is an irrational number, then there exist a geometric proof to its irrationality.*

Proof It is well known that every real number can be represented by a factorial series, see [3]. Hence

$$\theta = \frac{c_1}{1!} + \frac{c_2}{2!} + \dots + \frac{c_n}{n!} + \dots \quad (7)$$

where the $c_n (n = 1, 2, \dots)$ are integers, and moreover, $0 \leq c_n \leq n - 1$ for $n = 2, 3, \dots$. Note that the series in (7) is a Cantor series, where $a_n = c_{n+1}$ and $b_n = n + 1$. By a non-geometric proof to Cantor's theorem, see [4], $a_n > 0$ and $a_n < n$ hold infinitely often. So the geometric construction given in proof of theorem 1, gives a geometric proof to irrationality of θ . □

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